

## University of Dundee

### Dundee Discussion Papers in Economics 230

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*Publication date:*  
2010

[Link to publication in Discovery Research Portal](#)

*Citation for published version (APA):*

Chen, Y-F., & Zoega, G. (2010). *Dundee Discussion Papers in Economics 230: Strong hysteresis due to age effects*. (Dundee Discussion Papers in Economics; No. 230). University of Dundee.

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Working Paper  
No. 230  
Feb 2010  
ISSN:1473-236X

# Strong Hysteresis due to Age Effects

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February 2010

## ABSTRACT

Strong hysteresis in the labour market (see Cross, 1995) requires workers to be heterogeneous in terms of the cost of hiring and firing. We show how such heterogeneity arises naturally in labour markets due to differences in workers' age by showing that both the hiring and the firing thresholds for productivity are age dependent. The presence of strong hysteresis does not for this reason depend on ad-hoc differences in the cost of hiring and firing workers.

**Keywords:** Hysteresis, hiring and firing costs, heterogeneous workers.

**JEL Classification:** E24, J41, J64

Transitory shocks to labour demand may have a permanent effect on unemployment in which case there is hysteresis in unemployment as first suggested by Phelps (1972).<sup>1</sup> Cross (1993, 1995) describes hysteresis in the labour market that is caused by worker heterogeneity in terms of the cost of hiring and firing a worker. What has become known as *strong hysteresis* has many interesting properties when it comes to explaining the time path of unemployment. In this paper we propose a reason for the heterogeneity required for strong hysteresis by showing how workers' differences in terms of age cause differences in the cost of hiring and firing them.

Models exhibiting strong hysteresis have the property that hysteresis is an increasing function of the size of the initial change in unemployment. They can thus potentially explain one of the stylised facts of unemployment, documented by Bianchi and Zoega (1998) and Papell et al. (2000) that unemployment persistence is best captured by infrequent changes in mean unemployment. The kind of hysteresis exhibited by these models can be described as a particular type of a response of an input-output system when one modifies the value of an input. As described by Amable et al. (1991, 1992 and 1993), strong hysteresis has the following properties; the system is permanently affected when the value of the input is modified and brought back to its initial level; the history of past inputs matters; and the degree of persistence depends on the size of the shock. An input could in this case be changes in some component of aggregate demand, changes in the price of other inputs such as oil, or changes in real exchange rates.

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<sup>1</sup> The well-known hysteresis channels in this vein were discussed by Phelps (1972) and later developed by others; they include the insider-outsider model (see, amongst others, Lindbeck and Snower, 1989), models of human capital depreciation (see Layard, Nickell and Jackman, 1991) and models of physical capital depreciation (see Modigliani, 1987). The main weakness of these formulations is their prediction that all changes in unemployment, at least those with the same sign, should exhibit the same persistence. Empirically, hysteresis is sometimes estimated by using lagged values of unemployment to explain current unemployment (see Karanasou and Snower, 2007). But empirically the persistence can be better described as the outcome of large and infrequent shifts in the mean rate of unemployment.

The properties of the model are analogous to those of some well-known models of international trade, which emphasise heterogeneity across firms and sunk costs in generating hysteresis, the so-called beachhead effects of import penetration (see Baldwin, 1988; Dixit, 1989; Sim, 2006). Here real exchange rate depreciation makes exports rise and higher exports continue after the real exchange rate has appreciated again due to the sunk costs of advertising, acquiring customer loyalty, brand name recognition and goodwill. This idea has been applied to the labour market by Cross (1993) and Belke and Göcke (1993, 1999) who show how exchange rate uncertainty can create weak reactions in the labour market.

## **I. Hysteresis in unemployment**

A simple version of strong hysteresis requires there to be positive costs of hiring and firing and workers differing in the size of these costs. Hiring a new worker and firing an existing one entails sunk costs for this reason. One can then characterise any given worker by the two values of the aggregate productivity variable  $g$  at which the worker would be hired  $g_h$  and fired  $g_f$  respectively. Because both hiring and firing costs are non-negative we have  $g_h > g_f$ .

In Figure 1 below we show the two thresholds. When  $g < g_f$  the worker is unemployment since firms would not want to invest in hiring him and if he were employment they would invest in firing him due to the very low level of productivity. Similarly, when  $g > g_h$  the worker is employed because he would not be fired at this level of productivity and if he were unemployment firms would invest in hiring him. However in the range  $g_f < g < g_h$  the employment status of the worker would depend on the history of

shocks. If he was unemployed before, he would continue to be unemployed since firms would not want to invest in hiring him. However, if he was employed in the past, firms would want to keep him employed because productivity is not sufficiently low to justify investing in firing him. His employment status is hence history dependent.

However, while this simple setup generates what is termed *weak hysteresis* it is of limited use in describing unemployment dynamics because if all workers have identical threshold  $g_f$  and  $g_h$  we would find that they are either all employed, all unemployed or have an employment status that depends on history. For richer dynamics we need the thresholds to differ across workers.

When workers differ in the value of the thresholds, we can then show all workers in the  $g_h$ - $g_f$  space in Figure 2. Because  $g_h > g_f$  all workers will be found above the  $45^\circ$  line. The area can be divided into two spaces by a stepped line such as I-I. As we move to the right along the horizontal axis we find workers with lower firing costs, hence higher exit thresholds  $g_f$ . As we move down the vertical axis we find workers with lower hiring costs, hence lower entry thresholds  $g_h$ . The area below and to the left of this line,  $E$ , has employed workers and the area to the right of this line has unemployed workers,  $U$ . An increase in productivity – represented by a move up the  $45^\circ$  line – causes all workers with a lower  $g_h$  than the new realised productivity level to be hired. Similarly, a decrease in the aggregate productivity level would make all workers with a higher level of  $g_f$  be dismissed. Thus a sequence of changes in productivity creates a pattern of unemployment like the one in the figure below. The horizontal segments of this line correspond to past maxima of the productivity parameter and the vertical segments correspond to past minima. Here, the top horizontal line shows the maximum productivity level that was

reached in the past and the left-most vertical line shows the lowest minimum reached in the past and the right-most vertical line represents the highest local minimum. A movement up the  $45^\circ$  line gives workers with higher hiring costs and decreasing firing costs.

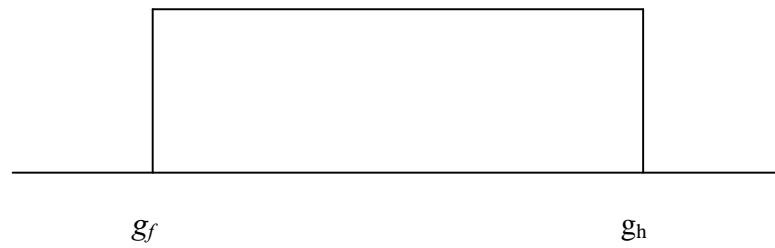
Due to worker heterogeneity in Figure 2 we find *strong hysteresis*; the system is permanently affected when the value of the input is modified and brought back to its initial level; the history of past inputs matters; and the degree of persistence depends on the size of the shock. Thus an increase in  $g$  can wipe out the effect of past maxima and a decrease can wipe out the effect of past minima. The history of past extrema which have not been exceeded by subsequent movements in  $g$  is preserved. In contrast to Phelps (1972), it is not the history of past unemployment rates that matters but the history of past changes in an “input” such as labour productivity. Moreover, it is the history of past extrema that matters.<sup>2</sup>

For the model to be plausible, we need to explain why workers should differ in terms of entry and exit thresholds. We will show below how such heterogeneity can arise even when the direct cost of hiring is identical for all workers and also the cost of firing simply because workers differ in terms of age, hence expected remaining job tenure. This is not the only reason for differences in the hiring and firing thresholds across workers. However, showing that differences in age suffice at generating strong hysteresis goes to show that such hysteresis arises quite naturally in labour markets. Our model extends the framework of Bentolila Bertola (1990) to the case of heterogeneous workers in terms of age while maintaining their assumption of linear adjustment costs.

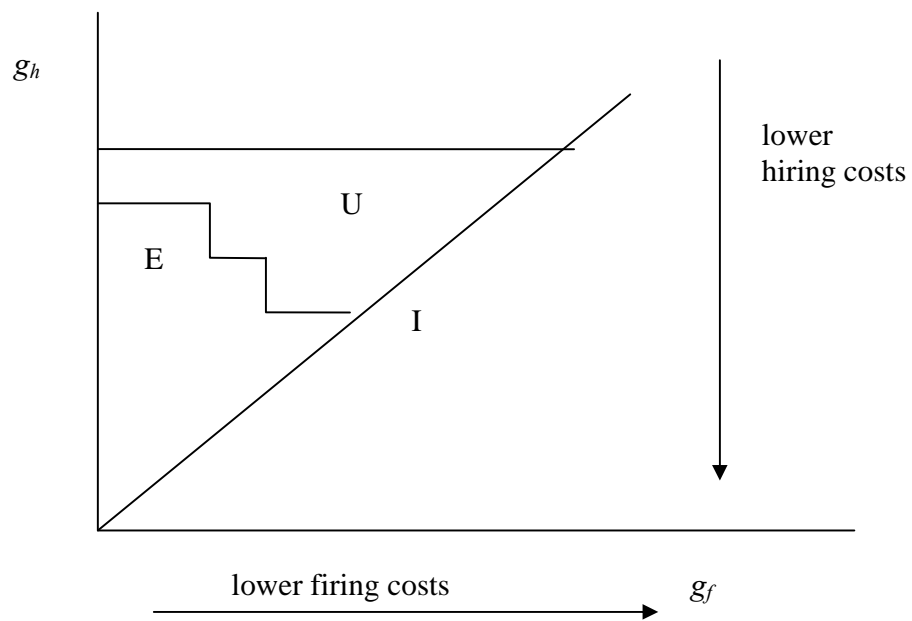
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<sup>2</sup> See Piscitelli et al. (2000) for empirical tests of strong hysteresis.

**Figure 1.** Entry and exit thresholds



**Figure 2.** The pattern of unemployment





## II. Age and heterogeneity

Differences in age may affect the decision by firms which workers to hire and which to fire. Firms facing temporary setbacks may want to fire the older workers due to their short remaining tenure and they may want to hire the younger workers when the recession is over because of their long expected tenure.<sup>3</sup> If so the young workers would be the first to be hired and the last to be fired, that is located close to the origin in Figure 2. Conversely, firms facing a long-term decline may choose to fire their younger workers first because they have the longest expected remaining job tenure.

In order to model the effect of age on firms' hiring and firing decisions we assume that all workers receive the same wage – wage differences are another reason why the thresholds might depend on age but we put that reason aside – but that workers differ in terms of expected remaining tenure. Firms are faced with fixed direct costs of hiring and firing each worker due to the cost of hiring and training, on the one hand, and severance payments and production disruptions, on the other hand. We assume that the direct costs of hiring and firing are independent of age so that the levels of training costs and severance pay are not increasing in age. Clearly, as in the case of wage differences, relaxing this assumption would also generate heterogeneity in terms of the thresholds but we will focus on the effect of differences in age without invoking any ad-hoc differences in hiring and firing costs.

We assume that the representative firm has current profits are defined as follows,

$$\Pi(g_t, N_t) = g_t N_t^\theta - w N_t, \quad 0 < \theta < 1, \quad (1)$$

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<sup>3</sup> Alternatively, as suggested by Lazear and Freeman (1997), firms may want to fire both the younger workers because their accumulation of job-specific skills has not been completed and the older workers because of short remaining tenure.

where  $N$  denotes the number of employed workers,  $w$  is the real wage, and  $g$  is a measure of productivity as before. It is assumed that each worker has a maximum working life of  $T$  at time zero ( $t=0$ ). Productivity  $g$  follows a standard geometric Brownian motion

$$dg_s = \eta g_s ds + \sigma g_s dW_s, \quad (2)$$

where  $W_s$  is a standard Wiener process,  $\eta$  is the drift parameter and  $\sigma$  the variance parameter. The quit rate of employees,  $\lambda$ , is assumed to be exogenous and identical across workers. The firm's expected marginal value of an employee without any firing and/or hiring is

$$v(Y, t; T) \equiv v(Y_t, t; T) = E \left[ \int_t^T (Y_s - w) e^{-(\rho + \lambda)(s-t)} ds \right], \quad (3)$$

where  $E[\cdot]$  is the expectation operator,  $\rho$  denotes the real interest rate,  $v$  is the (intertemporal) marginal value of workers and  $Y_s = \theta g_s N_s^{\theta-1}$  represents the marginal product of labour at time  $s$ . The corresponding Bellman equation for equation (3) is denoted by

$$(\rho + \lambda)v = Y - w + \eta_Y Y v_Y + \frac{1}{2} \sigma^2 Y^2 v_{YY} + v_t, \quad (4)$$

where  $\eta_Y = \eta + \lambda(1 - \theta)$  by the Ito's Lemma. As shown in Chen and Zoega (2009), the particular solution to (4) is denoted by

$$v^P = aY - bw, \quad (5)$$

where  $a = (1 - e^{-(\rho + \lambda - \eta_Y)(T-t)}) / (\rho + \lambda - \eta_Y)$ ,  $b = (1 - e^{-(\rho + \lambda)(T-t)}) / (\rho + \lambda)$  and it is assumed that the denominator of the parameter  $a$  is positive<sup>4</sup>. The options to hire ( $v_H^G$ ) and options

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<sup>4</sup> Alternatively, one can obtain equation (5) by integrating equation (3) directly without considering hiring and firing.

to fire ( $v_F^G$ ), from the homogenous part of equation (4), are represented by the following:<sup>5</sup>

$$v_H^G(Y, t; T) = A_1 Y^{\beta_1} N(d_1), \quad (6)$$

$$v_F^G(Y, t; T) = A_2 Y^{\beta_2} N(-d_2), \quad (7)$$

where  $A_1, A_2$  are unknown parameters,  $d_{1/2} = \frac{\ln Y \pm \sigma^2(T-t) \sqrt{\left(\frac{\eta_Y}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2(\rho+\lambda)}{\sigma^2}}}{\sigma \sqrt{T-t}},$

and  $N(d) = \left(1/\sqrt{2\pi}\right) \int_{-\infty}^d e^{-\varpi^2/2} d\varpi$ ,  $0 \leq N(d) \leq 1$ , is the cumulative normal distribution

function. Roots  $\beta$  are determined by equation (8)

$$\frac{1}{2} \sigma^2 \beta(\beta-1) + \eta_Y \beta - (\rho + \lambda) = 0, \quad (8)$$

where  $\beta_1$  and  $\beta_2$  are positive and negative roots of the above equation respectively.

The two marginal productivity thresholds for hiring and firing,  $Y_H$  and  $Y_F$  can be

obtained by the following value-matching and smooth pasting conditions

*Value-matching conditions*

$$aY_H - bw + v_F^G(Y_H, t; T, A_2) = H + v_H^G(Y_H, t; T, A_1), \quad (9)$$

$$-(aY_F - bw) + v_H^G(Y_F, t; T, A_1) = F + v_F^G(Y_F, t; T, A_2). \quad (10)$$

*Smooth-pasting conditions*

$$a + \frac{\partial v_F^G(Y_H, t; T, A_2)}{\partial Y_H} - \frac{\partial v_H^G(Y_H, t; T, A_1)}{\partial Y_H} = 0, \quad (11)$$

$$a + \frac{\partial v_F^G(Y_F, t; T, A_2)}{\partial Y_F} - \frac{\partial v_H^G(Y_F, t; T, A_1)}{\partial Y_F} = 0, \quad (12)$$

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<sup>5</sup> In the Appendix, we show that equations (6) and (7) are the solutions to the homogenous part of equation (4).

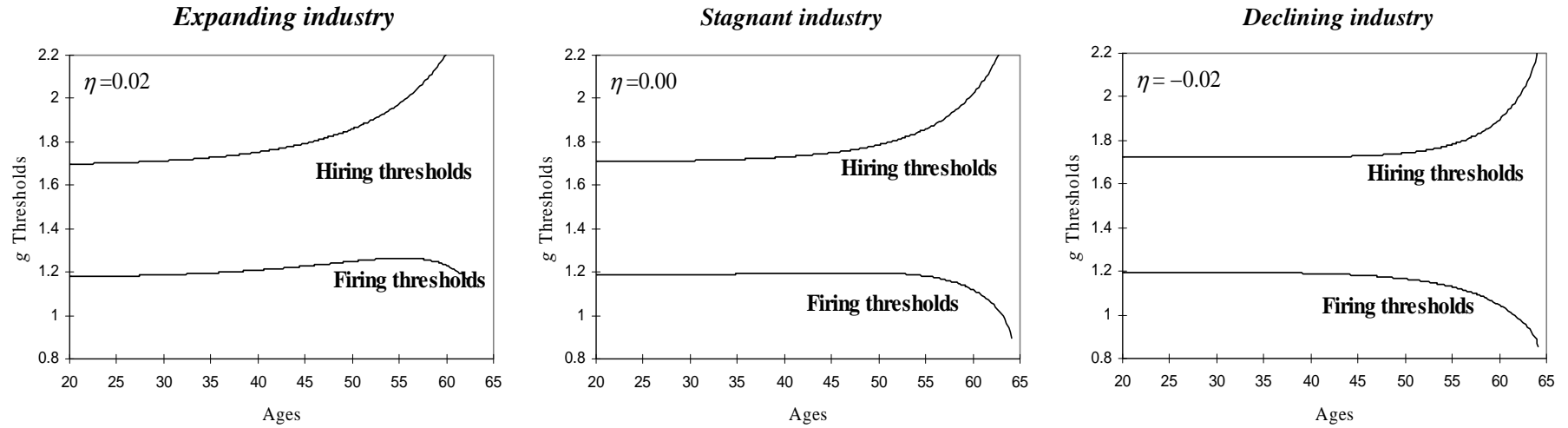
In the value-matching conditions, we find that the marginal benefit of hiring a worker consists of the sum of the particular solution – which we can take as the present discounted value of future profits – and the firing option, while the marginal costs consist of the sum of the hiring costs  $H$  and the sacrificed hiring option. Similarly, for the firing threshold, the marginal benefit of firing a worker consists of the sum of the negative of the particular solution – which is the present discounted value of future gains from firing which are equal to the negative of the present discounted value of future losses from employing the worker – and the hiring option, while the costs consist of the sum of the firing costs and the sacrificed firing option.

There are four unknown variables,  $Y_H$ ,  $Y_F$ ,  $A_1$ , and  $A_2$ , in four nonlinear equations and  $Y_H$  and  $Y_F$  are obtained accordingly. After obtaining the thresholds  $Y_H$ ,  $Y_F$ , we can then derive the thresholds for labour productivity for hiring and firing:  $g_H$  and  $g_F$  by applying the relationship  $Y = \theta g N^{\theta-1}$ . Thus, we have the hiring productivity thresholds

$$g_h = Y_H N^{1-\theta} / \theta \text{ and firing productivity thresholds } g_f = Y_F N^{1-\theta} / \theta .$$

Figure 3 shows the thresholds for three cases, first when productivity is rising but wages are constant – the case of an expanding industry – then when productivity is constant and wages are constant – the case of a stagnant industry – and finally the case when productivity is declining and wages constant – the case of a declining industry.

**Figure 3.** Hiring and firing thresholds



The effect of age on the hiring- and firing thresholds with different effective firing costs. Ages are equal to  $(65-T)$ . Other parameters:  $\sigma=0.20$ ,  $\rho=0.10$ ,  $\theta=0.7$ ,  $\lambda=0.05$ ,  $w=1$ ,  $H=0.083$ ,  $F=0.1$ ,  $N=1$ , and  $t=0$ .

Note: Productivity is growing at rate  $\eta=0.02$  while wages are constant in the expanding industry. Productivity and wages are both constant in the stagnant industry. Productivity declines at rate  $\eta=-0.02$  while wages are constant in the declining industry.

Both the hiring threshold and the firing threshold slope upwards in the figure for an expanding industry and a stagnant industry, although the slope of the firing threshold is much smaller in the stagnant industry. This implies that the productivity level at which a worker is hired is higher for the older workers – the case of a high hiring threshold  $g_h$  in Figure 2 above – and that the older workers will be fired at higher levels of productivity  $g_f$  – the case of a high firing thresholds in Figure 2. Older workers are thus more likely to be fired in a recession and not rehired in an ensuing recovery. A steep temporary recession in an otherwise growing economy is thus likely to leave a residue of older workers who remain unemployed until they retire from the labour force.

The intuition for the upward-sloping thresholds is easy to explain. Firms have the option of choosing when to hire new workers and to fire existing workers during economic downturns. Starting with the firing threshold, the option to choose when to fire workers has implications for the composition of the pool of workers fired. In a perfect-foresight framework management may decide in a downturn to fire its younger workers if it does not expect profits to recover – because the present discounted value of future losses from employing them exceeds that for the older workers because of longer expected tenure – while uncertainty about future productivity may convince management to wait before firing the young workers, the more so the greater the uncertainty. We call the first case the “tenure effect” and the second the “sacrificed options effect”. For moderate levels of firing costs, the sacrificed options effect dominates the tenure effect and firms choose to fire the older workers first. In a nutshell, firms hang on to the younger workers because it is more likely that productivity will recover during their remaining tenure. Thus with moderate firing costs the option to fire a worker is valueable

and the firm may hesitate to fire a given worker because productivity may improve in the future. The effect will be to protect the employment of young workers – who have long expected job tenure – at the expense of older workers. In contrast, older workers may be fired because it is less likely that a recovery may lead the firm management to regret such a decision in light of their short remaining tenure. Intuitively, at moderate levels of firing costs, the expected discounted losses are small at the firing margin, and a small recovery of productivity may turn losses into profits, the sacrificed option effect becomes stronger. This is more likely to happen in the case of a young worker. When considering the hiring decision in an expanding industry, in spite of the acquired firing option falling with age, firms always prefer to hire the younger workers first. This is due to the positive trend productivity growth which makes the firing option small in comparison to the present discounted value of future profits from hiring a worker.

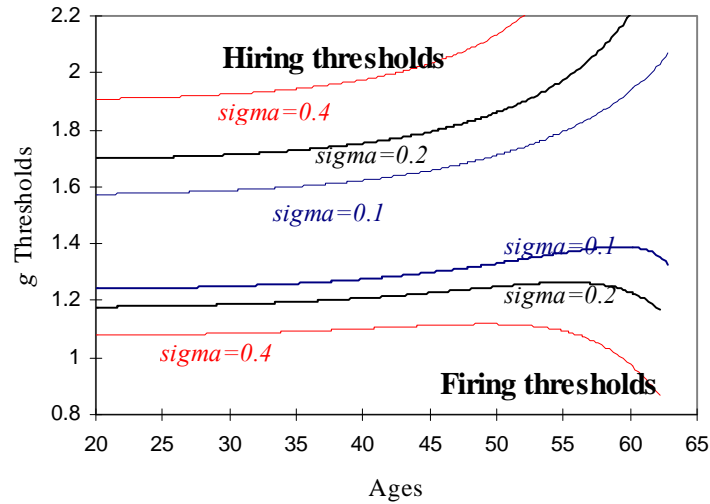
In the case of the declining industry – when the growth rate of  $g$  is negative and wages remain constant – the tenure effect is now much weaker for the hiring decision, stronger for the firing decision, and the hiring option more important than the firing option. This means that the tenure effect is dominated by the hiring option for hiring decisions and the old are hired first while the tenure effect dominates the firing options so that the young are the first to be fired. Firms are now more tempted to fire the younger workers because the benefit from waiting to see if productivity will recover is smaller due to the negative rate of growth of productivity. The slopes of the two thresholds are reversed in this case when compared to the case of an expanding industry.

In all cases – the expanding, the stagnant and the declining industry – the hiring option eventually becomes steeply upward sloping and the firing threshold becomes

steeply downward sloping as workers approach retirement. This signifies that firms are not willing to invest in either hiring or firing workers when they are close to retirement. The tenure effect dominates both hiring and firing options.

When the level of uncertainty is raised, the hiring threshold is shifted upwards and the firing threshold downwards so that the gap between the two of them is increased. This is shown in Figure 4 below for the case of an expanding industry.

**Figure 4.** Uncertainty and the thresholds

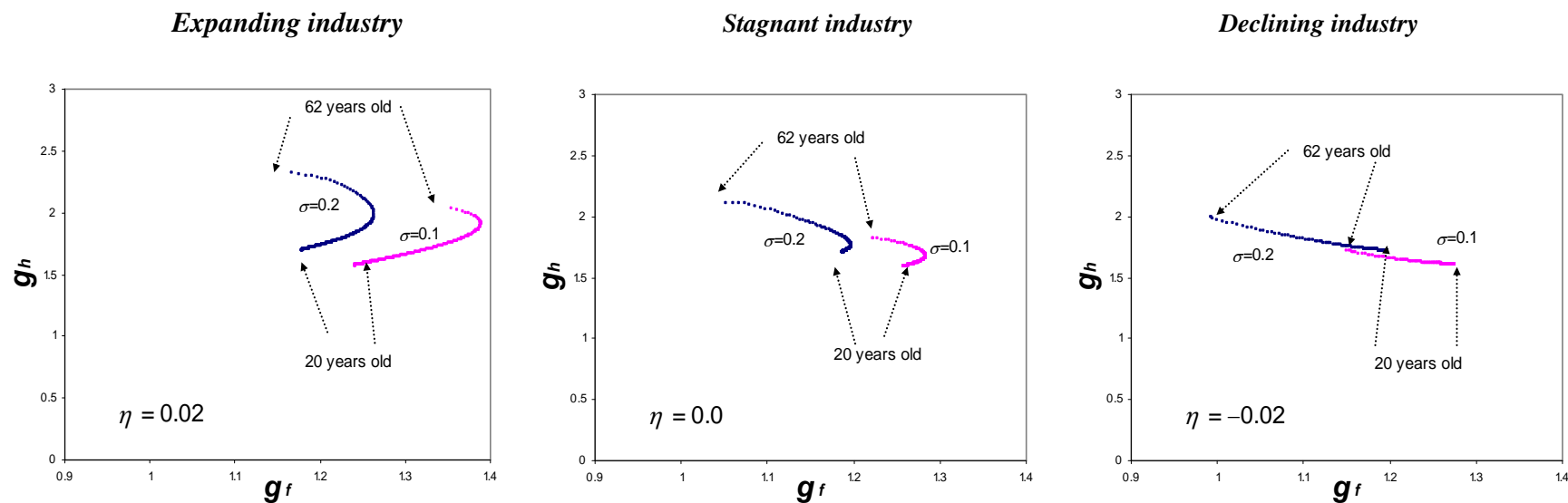


The effect of age on the hiring- and firing thresholds with different levels of uncertainty. Ages are equal to  $(65-T)$ . Other parameters:  $\rho=0.10$ ,  $\theta=0.7$ ,  $\eta=0.02$ ,  $\lambda=0.05$ ,  $w=1$ ,  $H = 0.083$ ,  $F=0.1$ ,  $N=1$ , and  $t=0$ .

Figure 5 below shows how the workers will be distributed in the  $g_h$ - $g_f$  space according to their age for the cases of an expanding industry, a stagnant industry and a declining industry for different levels of uncertainty. Clearly, differences in age explain why workers take different positions in Figure 2 above.



**Figure 5.** Firing and hiring thresholds for workers belonging to different age groups



The effect of age on the hiring- and firing thresholds belonging to different age groups. Ages are equal to  $(65-T)$ . Other parameters:  $\sigma=0.20$ ,  $\rho=0.10$ ,  $\theta=0.7$ ,  $\lambda=0.05$ ,  $w=1$ ,  $H=0.083$ ,  $F=0.1$ ,  $N=1$ , and  $t=0$ .

Note: Productivity is growing at rate  $\eta = 0.02$  and wages are constant in the expanding industry. Productivity and wages are both constant in the stagnant industry. Productivity declines by  $\eta = -0.02$  while wages are constant in the declining industry.

The reflection point for  $\sigma = 0.2$  is 55.2 years of age; for  $\sigma = 0.1$  it is 59.2 years in the expanding industry.  
 The reflection point for  $\sigma = 0.2$  is 47.4 years of age; for  $\sigma = 0.1$  it is 54.0 years in the stagnant industry.  
 No reflection points for the expanding industry, as all starting from age=20

With the passage of time, each worker will move along one of the trajectories shown in Figure 5 above. A worker employed in an expanding or a stagnant industry will initially both face rising prospects of being fired – a higher firing threshold – and also dimmer prospects when it comes to reemployment – a rising hiring threshold – but then after the age of 55 (for  $\sigma = 0.2$ ) in the expanding industry and after the age 47 in the stagnant industry he will face a falling probability of being fired – a falling firing threshold – but also ever falling chances of being rehired following a dismissal – rising hiring threshold. In the declining industry, he will only face a falling probability of being fired throughout his tenure and dimmer prospects when it comes to the probability of being rehired following a dismissal.

#### **IV. Conclusions**

A model exhibiting strong hysteresis can potentially explain an important stylised fact regarding post-war unemployment in the OECD countries, which is the positive relationship between the persistence of a transitory shock to unemployment and its size. Persistently high unemployment arises following large shocks such as the oil price shocks in the 1970s and the severe recession at the beginning of the 1980s.

Models of strong hysteresis have used the ad-hoc assumption that workers are heterogeneous in terms of the cost of hiring and firing. We have shown that such heterogeneity arises quite naturally due to differences in workers' age by showing how differences in age – hence expected remaining job tenure – can generate differences in the level of productivity at which each worker is hired and fired. For this reason we expect strong hysteresis to arise quite naturally in labour markets.

## Appendix: Derivation of Equations (5) and (6)

The homogenous part of equation (4) in the text has the following form:

$$(A1) \quad (\rho + \lambda)v^G = \eta_Y Y v_Y^G + \frac{1}{2} \sigma^2 Y^2 v_{YY}^G + v_t^G.$$

As Chen and Zoega (2009) has shown the details derivations for the solution for equation (A1), here we use another way to show that the hiring options,  $v_H^G(Y, t; T) = A_1 Y^{\beta_1} N(d_1)$ , and then the firing options, are the true solutions to (3). Note that the positive  $\beta_1$  has the form from equation (8) in the text:

$$(A2) \quad \beta_1 = \frac{1}{2} - \frac{\eta_Y}{\sigma^2} + \sqrt{\left(\frac{\eta_Y}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2(\rho + \lambda)}{\sigma^2}} > 0.$$

$$\text{And } d_1 = \frac{\ln Y + \sigma^2(T-t) \sqrt{\left(\frac{\eta_Y}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2(\rho + \lambda)}{\sigma^2}}}{\sigma \sqrt{T-t}}, \quad N(d) = \left(1/\sqrt{2\pi}\right) \int_{-\infty}^{d_1} e^{-\varpi^2/2} d\varpi.$$

Differentiation  $v_H^G(Y, t; T) = A_1 \left(1/\sqrt{2\pi}\right) Y^{\beta_1} \int_{-\infty}^{d_1} e^{-\varpi^2/2} d\varpi$  by using Leibnitz rule gives

$$(A3) \quad \eta_Y Y v_Y^G = \eta_Y \left[ \beta_1 Y^{\beta_1} \int_{-\infty}^{d_1} e^{-\varpi^2/2} d\varpi + Y^{\beta_1} \frac{1}{\sigma \sqrt{T-t}} e^{-d_1^2/2} \right] \frac{A_1}{\sqrt{2\pi}},$$

$$(A4) \quad v_t^G = \left[ \frac{\ln Y}{2\sigma(T-t)\sqrt{T-t}} - \frac{\sigma}{2\sqrt{T-t}} \sqrt{\left(\frac{\eta_Y}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2(\rho + \lambda)}{\sigma^2}} \right] \frac{A_1}{\sqrt{2\pi}} Y^{\beta_1} e^{-d_1^2/2},$$

$$\frac{1}{2} \sigma^2 Y^2 v_{YY}^G = \frac{\sigma}{2\sqrt{T-t}} \beta_1 \frac{A_1}{\sqrt{2\pi}} Y^{\beta_1} e^{-d_1^2/2} + \frac{\sigma^2}{2} \beta_1 (\beta_1 - 1) \frac{A_1}{\sqrt{2\pi}} Y^{\beta_1} \int_{-\infty}^{d_1} e^{-\varpi^2/2} d\varpi$$

$$(A5) \quad + \frac{1}{2(T-t)} \frac{A_1}{\sqrt{2\pi}} Y^{\beta_1} \left[ \frac{- \left[ \ln Y + \sigma^2(T-t) \sqrt{\left(\frac{\eta_Y}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2(\rho + \lambda)}{\sigma^2}} \right]}{\sigma \sqrt{T-t}} \right] e^{-d_1^2/2}$$

$$+ \frac{\sigma}{2\sqrt{T-t}} (\beta_1 - 1) \frac{A_1}{\sqrt{2\pi}} Y^{\beta_1} e^{-d_1^2/2}.$$

Substituting (A3) – (A5) back to equation (A1) and collecting terms give

$$\begin{aligned}
& \left[ \frac{\sigma^2}{2} \beta_1 (\beta_1 - 1) + \eta_Y \beta_1 - (\rho + \lambda) \right] \frac{A_1}{\sqrt{2\pi}} Y^{\beta_1} \int_{-\infty}^{d_1} e^{-\varpi^2/2} d\varpi \\
\text{(A6)} \quad & + \left[ \beta_1 - \frac{1}{2} + \frac{\eta_Y}{\sigma^2} - \sqrt{\left( \frac{\eta_Y}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2(\rho + \lambda)}{\sigma^2}} \right] \frac{\sigma}{\sqrt{T-t}} \frac{A_1}{\sqrt{2\pi}} Y^{\beta_1} e^{-d_1^2/2} = 0
\end{aligned}$$

The items in two brackets are equal to zero due to equation (8) in the text and equation (A2), which concludes the proof that  $v_H^G(Y, t; T) = A_1 Y^{\beta_1} N(d_1)$  is the general solutions. The options to fire can be shown in a similar way.

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